ON EQUILIBRIUM CONFIGURATIONS OF SUPERDENSE DEGENERATE GAS MASSES

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Equilibrium densities of stellar masses possessing a density of the same order as the density of the atomic nucleus, and consisting of a highly degenerate baryon gas, are discussed. This treatment differs from that of Oppenheimer and Volkoff [4] in that the presence of hyperons at high densities is considered. The treatment is based on the Einstein gravitational equations. The possible existence of some equilibrium configurations containing a large number of hyperons is pointed out. The equilibrium models are computed for both and "ideal" Fermi gas and a real gas in which interaction between particles is taken into account.

I. Introduction

It is now generally known [1, 2] that complex atomic nuclei are incapable of existence at densities of $\rho > 10^6$ g/cm^3. When the density of matter rises to this value, the totality of nuclei present transforms to an assemblage of simple nucleons. If the temperature is so low that electrons and nucleons are degenerate, then further increase in density to $\rho \sim 3 \cdot 10^{11}$ g/cm^3 will initiate a numerical preponderance of neutrons over protons and electrons, and at $\rho \sim 10^{12}$ g/cm^3 the neutrons will begin to predominate with respect to the pressure they exert [3]. Thus, the superdense stellar masses at $\rho \gtrsim 10^{12}$ g/cm^3 must be composed predominantly of neutrons.

The equilibrium configurations of neutronic stars have been studied by Oppenheimer and Volkoff [4]. They showed that the masses of neutronic stars must have values falling in the range $0.3\odot < M_g \lesssim 0.7\odot$, while the radius values must lie within the range $6 \lesssim R \lesssim 20$ km. This is the fundamental finding of their work. It is assumed, in their calculations, that the neutrons form an ideal gas (naturally, a degenerate gas) all the way to as high densities as we please.

Some further development of the question was achieved in a recent contribution by Cameron [5]. He took account of the forces of mutual repulsion between neutrons acting at close range. As a result, it was found that the masses of some configurations of neutronic stars may reach $2\odot$. Cameron, independently of the authors of the present article, noted that hyperons must necessarily appear in degenerate stars at fairly high densities. However, he did not publish any further probings into this matter.

The present article is a continuation of our previous contribution [6] in which the question of the appearance of hyperons at high densities in a degenerate gas was investigated, and the effect of hyperons on the equation of state of the latter was discussed.

Let us recall some of the fundamental findings reported in that paper.

At a baryon density $N \geq 6.4 \cdot 10^{38}$ cm^{-3} (i.e., at $\rho \geq 1.1 \cdot 10^{15}$ g/cm^3), highly degenerate matter must inevitably contain hyperons and μ^--mesons in addition to nucleons. At still higher densities, π^--mesons must also make their appearance. Therefore, at densities of matter falling within the range $10^{12} \lesssim \rho \lesssim 10^{15}$ g/cm^3, we are dealing with a neutron gas, and in the range $\rho > 10^{15}$ g/cm^3, with a gas constituting a mixture of hyperons and nucleons, the percentage of hyperons increases rapidly with the density.

The concentrations of various particles in an equilibrium degenerate gas are determined from the following equations which contain the critical Fermi energies E and, accordingly, the concentrations of different species of particles:

$$\left.\begin{aligned} &E_{Y^0} = E_n; \qquad E_{Y^+} = E_n - E_e; \\ &E_{Y^-} = E_n + E_e; \quad E_e = E_\mu = m_\pi c^2; \\ &\Sigma N_{Y^+} - \Sigma N_{Y^-} - N_e - N_\mu - N_\pi = 0, \end{aligned}\right\} \quad (1.1)$$

where the symbols Y^0, Y^+, Y^-, n, e, μ, and π signify, respectively, neutral, positive, and negative baryons (general terms covering nucleons and hyperons), neutrons, electrons, and μ-mesons and π-mesons.

To each particle corresponds some threshold density value starting from which the particle is capable of existence in a medium as a stable component of matter. The equations cited fully determine the concentrations of all the particles, given the total number of baryons per unit volume. However, a system consisting of only those equations which are compatible with a given concentration must be taken for each value of the concentration of all baryons, while the concentrations of those baryons which do not become included in the system of equations in this case must then be set equal to zero.

Solving Eqs. (1.1), we find the concentrations of the particles. These concentrations may be expressed in terms of the parameters:

$$t_k = 4\,\mathrm{arsh}\,(p_k/m_k c), \tag{1.2}$$

where m_k is the mass of the k-th particle, p_k is the critical Fermi momentum,

$$p_k = (6\pi^2/a_k)^{1/3} h N_k^{1/3} \tag{1.3}$$

for the k-th particle, $a_k = 2S_k + 1$ is the number of particle spin states.

The parameters t_k, in the case of a spherical star, must be functions of the distance r from the center of the star. Various t_k values are then expressed in terms of t_n by the following relations.

In the case of neutral particles, i.e., at k = n*, Λ, Σ° and Ξ°

$$t_k = 4\,\mathrm{arch}\left(\frac{m_n}{m_k}\,\mathrm{ch}\,\frac{t_n}{4}\right), \tag{1.4}$$

and in the case of positive baryons, i.e., at k = p, p*, Σ^+,

$$t_k = 4\,\mathrm{arch}\left(\frac{m_n}{m_k}\,\mathrm{ch}\,\frac{t_n}{4} - \frac{E_e}{m_k c^2}\right), \tag{1.5}$$

and, finally, in the case of negative baryons, i.e., at k = Σ^- and Ξ:

$$t_k = 4\,\mathrm{arch}\left(\frac{m_n}{m_k}\,\mathrm{ch}\,\frac{t_n}{4} + \frac{E_e}{m_k c^2}\right), \tag{1.6}$$

π^--mesons exist in a star only at a baryon concentration $N > 5.9 \cdot 10^{40}$ cm^{-3}, i.e., $\rho > 1.4 \cdot 10^{17}$ g/cm^3. At fairly high densities, there exists no essential difference between the right-hand members of Eqs. (1.4), (1.5), and (1.6). Since $E_e \leq m_\pi c^2$, this always occurs when

$$\mathrm{ch}\,\frac{t_n}{4} \gg \frac{m_\pi}{m_n}.$$

II. Fundamental Equations of Equilibrium Configurations

In the case of stellar configurations of the conventional type (for example, for our sun), the actual radius of the star is very large compared to its gravitational radius. In the case of superdense configuration states, these two quantities become comparable. This means that it is not permissible to neglect the effects of the general theory of relativity. In other words, these calculations must be carried out with Einstein's gravitational equations as the basis.

We shall seek to find the solution to Einstein's equations for the case of a spherically symmetric distribution of matter. It is known that an arbitrary selection of coordinate systems may be employed within the framework of the general theory of relativity such that the four-dimensional space will be of the form [7]:

$$ds^2 = c^2 e^\nu dt^2 - r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) - e^\lambda dr^2, \tag{2.1}$$

where ν and λ are functions of r. In each concrete problem, these quantities are subject to redefinition. Formula (2.1) is valid not only for a steady-state distribution of matter in the star, but also in the case where radial motions occur in the star, velocity being a function of r and t alone, so that the spherical symmetry of the distribution of matter is unaffected. Of course, the functions ν and λ are dependent not only on r in this case, but also on t. Oppenheimer and Volkoff [4] have shown that the solution of Einstein's equations in the static case boils down to the solution of the following comparatively simple differential equations:

$$\frac{du}{dr} = 4\pi \frac{1}{c^2} r^2 \rho; \tag{2.2}$$

$$\frac{dP}{dr} = -\frac{P + \rho}{r\left(\frac{c^2}{k} r - 2u\right)}\left(4\pi \frac{1}{c^2} r^3 P + u\right), \tag{2.3}$$

where P (r) is the pressure, k is the gravitational constant, and u (r) is defined by the formula

$$u(r) = \frac{c^2 r}{2k}(1 - e^{-\lambda}). \tag{2.4}$$

At the boundary of the star, where r = R, the value of u (R) is equal to the mass of the star as perceived by an outside observer. As regards the significance of u (r) for small values of r, it is clear from Eq. (2.2) that this value in a certain sense characterizes the quantity of matter enclosed within a sphere of radius r. However, under the conditions considered, when the gravitational field is highly intense and the gravitational mass defect may be large, the concept "mass enclosed within a sphere of radius r" is required in the strict sense. We shall not dwell on that matter at this point. For our purposes, the definition (2.4) will suffice.

Later on, we shall use the system of units for which [4].

$$\frac{m_n^4 c^5}{32\pi^2 h^3} = K_n = \frac{1}{4\pi}; \quad c = k = 1. \tag{2.5}$$

Masses and distances have the same dimensionality in these units.

The unit of distance in this system is

$$a = 2\sqrt{2\pi}\left(\frac{h}{m_n c}\right)^{3/2} \frac{c}{\sqrt{k m_n}} = 1.37 \cdot 10^6 \text{ cm} \tag{2.6}$$

The unit of mass is

$$b \equiv \alpha \frac{c^2}{k} = 1.85 \cdot 10^{34} \mathrm{g} = 9.29 \odot. \quad (2.7)$$

Finally, the unit of time is

$$\delta \equiv \alpha / c = 4.57 \cdot 10^{-5} \text{ sec.} \quad (2.8)$$

In this system of units, Eqs. (2.2) and (2.3) transform to:

$$\frac{du}{dr} = 4\pi r^2 \rho(r); \quad (2.2')$$

$$\frac{dP}{dr} = -\frac{P+\rho}{r(r-2u)}(4\pi r^3 P + u). \quad (2.3')$$

Equations (2.2) and (2.3) contain three unknown functions ρ, P, and $\underline{u}$. The equation of state, about which more will be said in the following section, should be included with the above as a third equation.

Solving this system of three equations, we find a family of solutions each of which corresponds to a certain mass, radius, internal distribution of matter, and space metric.

Integration was carried out by the Runge-Kutta method for several variants of the equation of state.

III. Configuration Consisting of an Ideal Gas

For an ideal degenerate gas, the equation of state appears in parametric form as follows [6]:

$$\rho = K_n \sum_k \frac{1}{2} \alpha_k \left(\frac{m_k}{m_n}\right)^4 (\mathrm{sh}\, t_k - t_k) + N_\pi m_\pi c^2; \quad (3.1)$$

$$P = \frac{1}{3} K_n \sum_k \frac{1}{2} \alpha_k \left(\frac{m_k}{m_n}\right)^4 \left(\mathrm{sh}\, t_k - 8\, \mathrm{sh} \frac{t_k}{2} + 3t_k\right), \quad (3.2)$$

N_π is the density of π^--mesons. The definition of K_n is found in Eq. (2.5). Summation is carried out over all species of particles present at the given point in the star. All parameters t_k (i.e., also concentrations N_k), as well as N_π, are functions of the density of matter. It is convenient to choose a parameter t_n, i.e., a concentration N_n, as the independent variable in the calculations. Equations (3.1) and (3.2) are expressed in CGS units. In the system of units used in Eq. (2.5), K_n is to be replaced by $1/4\pi$.

Substitution of Eqs. (3.1) and (3.2) into (2.2') and (2.3') yields

$$\frac{du}{dr} = r^2 \left[\sum_k \frac{1}{2} a_k \left(\frac{m_k}{m_n}\right)^4 (\mathrm{sh}\, t_k - t_k) + N_\pi m_\pi c^2 / K_n\right]; \quad (3.3)$$

$$\frac{dt_n}{dr} = -\frac{4}{r(r-2u)} \frac{\sum \frac{1}{2} a_k \left(\frac{m_k}{m_n}\right)^4 \left(\mathrm{sh}\, t_k - 2\, \mathrm{sh} \frac{t_k}{2}\right) + 3N_\pi m_\pi c^2 / 4K_n}{\sum_k \frac{1}{2} a_k \left(\frac{m_k}{m_n}\right)^4 \left(\mathrm{ch}\, t_k - 4\, \mathrm{ch} \frac{t_k}{2} + 3\right) \frac{\partial t_k}{\partial t_n}} \times \left\{u + \frac{1}{3} r^3 \sum_k \frac{1}{2} a_k \left(\frac{m_k}{m_n}\right)^4 \left(\mathrm{sh}\, t_k - 8\, \mathrm{sh} \frac{t_k}{2} + 3t_k\right)\right\}. \quad (3.4)$$

In sum, we have to integrate Eqs. (3.3) and (3.4) in order to ascertain the internal structure of a star. We might also proceed directly from Eqs. (2.2') and (2.3'). In the latter case, it would be feasible to first plot a graph of $\rho = \rho(P)$, i.e., a curve of state, on the basis of Eqs. (3.1) and (3.2).

We must start the integration of Eqs. (3.3) and (3.4) from the center of the star, where

$$u(0) = 0; \quad t_n = t_n(0). \quad (3.5)$$

$t_n(0)$ here defines the density of matter at the center of the star. The various values $t_n(0)$ then corresponds to the various configurations. In practice, we obtain a number of solutions specified by several arbitrary assignments of $t_n(0)$. Once we find these solutions, by the approach to be described below, we will be able to determine, for each of them, the values of the observed parameters, e.g., radius or mass of the particular configuration. The observed parameters are arrived at in this way as functions of $t_n(0)$. After eliminating $t_n(0)$, we are in a position to determine each of the others. In particular, this approach leads us to the relationship between mass and radius for our family of configurations. With reference to the first of the initial conditions (3.5), this means that there is no point mass at the center of the star.

In the case where we employ Eqs. (2.2') and (2.3') directly, the initial conditions will have the form:

$$u(0) = 0; \ \rho = \rho(0); \ P = P(\rho(0)). \quad (3.6)$$

where $\rho(0)$ is the density at the center of the particular configuration.

TABLE 1. Several of the Most Important Parameters of Hyperonic Stars Consisting of an Ideal Gas of Baryons

$t_n(0)$	Star						Hyperonic core of star			
	central density		mass		coordinate radius		mass		coordinate radius	
	Baryons $N(0)$, cm^{-3}	Matter $4\pi\cdot\rho(0)$, g/cm^3	units of Eq. (2.5)	solar mass units	units of Eq. (2.5)	km	units of Eq. (2.5)	solar mass units	units of Eq. (2.5)	km
0.566	$1.00\cdot10^{37}$	$1.62\cdot10^{13}$	0.0147	0.136	2.090	28.7	—	—	—	—
1.0	$6.50\cdot10^{37}$	$1.00\cdot10^{14}$	0.0329	0.306	1.537	21.1	—	—	—	—
1.3	$1.36\cdot10^{38}$	$2.24\cdot10^{14}$	0.0443	0.411	1.345	18.4	—	—	—	—
1.5	$2.20\cdot10^{38}$	$3.62\cdot10^{14}$	0.0495	0.460	1.170	16.0	—	—	—	—
1.8	$3.85\cdot10^{38}$	$6.60\cdot10^{14}$	0.0600	0.557	1.065	14.6	—	—	—	—
2.4	$1.25\cdot10^{39}$	$2.34\cdot10^{15}$	0.0683	0.634	0.805	11.0	0.0132	0.123	0.245	3.36
2.75	$3.1\cdot10^{39}$	$5.92\cdot10^{15}$	0.0622	0,578	0.753	10.3	0.0234	0.217	0.270	3.20
3.0	$5.45\cdot10^{39}$	$1.09\cdot10^{16}$	0.0559	0.519	0.703	9.63	0.0255	0.237	0.254	3.48
3.3	$1.08\cdot10^{40}$	$3.62\cdot10^{16}$	0.0484	0.450	0.670	9.18	0.0258	0.240	0.232	3.18
4.0	$3.50\cdot10^{40}$	$8.29\cdot10^{16}$	0.0354	0.329	0.594	8.14	0.0225	0.209	0.185	2.54
5.0	$1.26\cdot10^{40}$	$3.45\cdot10^{17}$	0.0245	0.228	0.506	6.93	0.0173	0.161	0.137	1.88
6.0	$3.53\cdot10^{41}$	$1.15\cdot10^{18}$	0.0191	0.177	0.557	7.63	0.0135	0.125	0.115	1.57
7.0	$9.36\cdot10^{41}$	$3.39\cdot10^{18}$	0.0237	0.220	0.767	10.5	0.0115	0.107	0.112	1.53
∞	∞	∞	0.0349	0.324	0.808	11.1	0.0130	1.121	0.145	1.99

Proceeding from the initial conditions, integration is carried out step by step all the way to $r = R$, where R is defined from the condition $\rho(R) = P(R) = t_n(R) = 0$. Then R will signify the coordinate radius of the configuration, and $u(R) \equiv M$ will be its mass.

The mass of the region containing hyperons, i.e., the mass of the hyperon core, will be of interest as a characteristic property of the stellar configuration. Hyperons are capable of existing in an equilibrium state beginning with $t_n = 2.1$. The value of R_h at which $t_n(R_h) = 2.1$ will consequently be the coordinate radius of the hyperon sphere. As regards the mass concentrated in this sphere, the value of $u(R_h)$, which we here denote as M_h, will be a measure of this quantity. Results of pertinent computations appear in Table 1. Values of coordinate radius and mass both for the star in its entirety and for the hyperon core, with the mass of the hyperon core arbitrarily understood in the sense alluded to above, are tabulated there alongside entries giving data on the concentrations of baryons and density values at the center of the star. The first five models (first five rows of the table) do not contain hyperon cores. They are purely neutron stars. The data on these five lines agree well with the findings reported by Oppenheimer and Volkoff. Starting with the value of central density corresponding to $t_n(0) = 2.1$, the characteristics of our configurations diverge from the characteristics for purely neutron configurations.

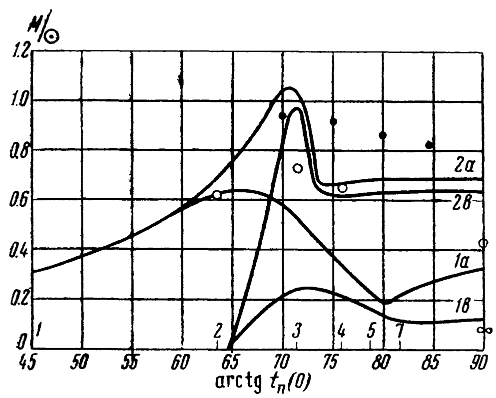

Fig. 1. Plot of mass of star as a function of $t_n(0)$ defining the density of matter at the center of the star. The definition of t_n is given in Eq. (1.2). Curves 1a and 1b depict the mass of the star and its hyperon core in the case where elementary particles form an ideal gas at any densities. Curves 2a and 2b present the same picture for the case where repulsive forces acting between baryons at high densities (real gas) are taken into account. Black dots indicate the mass of configurations consisting of neutrons, with repulsive forces taken into account. Hollow dots indicate the mass of those configurations consisting of an ideal neutron gas (calculations by Oppenheimer and Volkoff). The mass is expressed in units of solar mass.

In Fig. 1, curve 1a, depicts the dependence of the mass of the configuration on $t_n(0)$. For purposes of comparison, hollow dots are added to indicate the results of Oppenheimer and Volkoff, where the existence of hyperons was left out of account. As is clear from the graph, a decrease in the mass of a configuration is arrived at in calculations taking hyperons into account.

Curve 1b, also in Fig. 1, represents the mass of the hyperon core in the sense alluded to above. As is readily seen from the graph, a significant portion of the configuration mass is contained within the hyperon core when at $3 < t_n(0) < 7$. The value of a star's mass hits a minimum at $t_n(0) \approx 6$, corresponding to a baryon concentration of the order of $3.5 \cdot 10^{41}\,\mathrm{cm}^{-3}$. At those density levels, a degenerate gas of baryons would become relativistic.

In Fig. 2, curve 1a, shows the relationship of coordinate radius and $t_n(0)$, while curve 1b shows the relationship between the coordinate radius of a hyperonic core and the same quantity.

Figure 3 shows curve 1 as the relation between coordinate radius and mass for the configurations calculated in the present paper.

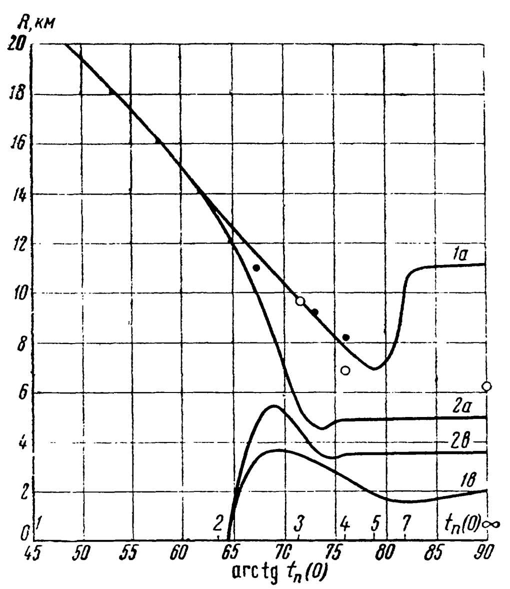

Fig. 2. Dependence of coordinate radius of star and coordinate radius of its hyperonic core on the parameter $t_n(0)$. The legend is same as in Fig. 1. The radius is expressed in kilometers.

IV. Case of Infinite Density at the Center

In the general case, numerical integration of our equations does not meet with any severe complication when the density at the center is a finite quantity. It would be of interest, however, to consider the limiting case when $t_n(0) = \infty$. As we see from Fig. 3, the radius and mass then tend to finite limits. Some limiting configuration for which $\rho(0) = \infty$ must then clearly exist. At very high densities, a baryon gas becomes extremely relativistic. At such density levels, we may assume $P \approx \rho/3$. However, at this point we shall use

$$P = z\rho, \tag{4.1}$$

since later on we shall pursue the same reasoning for the alternative case when $z \neq 1/3$. Now, for the case $z =$

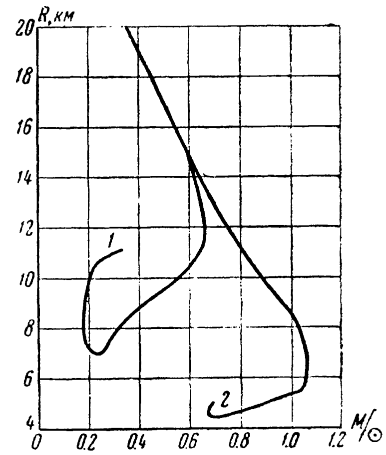

Fig. 3. Relationship between mass and radius for baronic stars. Curve 1 refers to those configurations consisting of a real baryon gas. The radius is taken to mean the value of the coordinate r at the surface of the star.

$1/3$ which we entertain, Eq. (4.1) is fulfilled approximately starting our from the center to those distances at which a baryon gas ceases to be highly relativistic.

Within this central sphere $t_n(r) \gg 1$, and the solution to Eqs. (2.2') and (2.3') may be sought in the following form:

$$4\pi \cdot \rho(r) = \frac{a}{r^n}. \tag{4.2}$$

Substitution of Eqs. (4.2) and (4.1) into Eqs. (2.2') and (2.3') yields forthwith:

$$n = 2;\ a = \frac{2z}{4z + (1+z)^2}. \tag{4.3}$$

As a result, we obtain the solution in the form:

$$4\pi\rho = \frac{a}{r^2};\ u(r) = ar. \tag{4.4}$$

In particular, at $z = 1/3$, we have $a = 3/14$:

$$4\pi \cdot \rho(r) = \frac{3}{14}\frac{1}{r^2};\ u(r) = \frac{3}{14}r. \tag{4.5}$$

We accepted the solution (4.5) out to the distance $r = 6.9 \cdot 10^{-4}$. At that distance we have $\rho = 3.58 \cdot 10^4$; $P = 1.14 \cdot 10^4$; and $u = 1.48 \cdot 10^{-4}$. All the values here appear in the units of Eq. (2.5). Assuming these values for r, ρ, P, and u to be the initial conditions, Eqs. (2.2') and (2.3') were then integrated numerically to the periphery of the star, i.e., to the distance where $\rho = P = 0$. In the course of the numerical solution, it was found that the asymptotic solution which we found retained its validity even for values of r far exceeding $6.9 \cdot 10^{-4}$, i.e., the value of r at which we began the numerical integration. For example, at the distance $r = 5.5 \cdot 10^{-3}$, the value of u calculated according to the asymptotic solution deviated by only 1% from the exact solution.

As is clearly seen from the last line in Table 1, the limiting mass corresponding to the solution treated here is $0.324\odot$, and R = 11.1 km.

The corresponding configuration in the theory proposed by Oppenheimer and Volkoff exhibits a mass $0.43\odot$ and a radius 6.2 km. It is clear that the values $M = 0.34\odot$ and R = 3.1 km reported by Oppenheimer and Volkoff. are in error.

V. Configuration of a Nonideal Baryon Gas

An analysis of experiments related to a study of collisions between energetic nucleons has shown that at small interparticle distances of the order of 0.4 fermi or less, extremely intense repulsive forces apparently come into action. These forces are sometimes approximated in the form of the Dirac delta function, i.e., it is assumed that the nucleon has a hard core with a radius of the order of 0.2 fermi. We infer from the preceding remarks that a baryonic gas can no longer be considered an ideal gas at at baryon density $N \gtrsim 10^{40}$ cm^{-3} ($t_n \gtrsim 3.2$) when the mean interparticle distance $l < 0.5$ fermi.

Unfortunately, the behavior of nuclear forces at high particle speeds is known only to a slight extent. It is evident that these forces are essentially dependent not only on distance but on particle velocities and spins. However, we still have no rigorous description of the interaction between nucleons at high velocities. As pertains to hyperons, nothing whatever is known about their interactions. It can only be said that interactions between hyperons are also strong interactions (bringing nuclear forces into play).

Such being the situation, it is of course difficult to construct any rigorous theory of superdense configurations taking interaction between baryons into account. It is however possible to consider, in a rough way, the effect of this interaction in the equation of state, so as to determine the direction and order of variation of parameters (mass, radius, and others) characterizing the star. It is evident that the introduction of repulsive forces between baryons at close range leads to increased internal pressure, and as a corollary to increased masses of the configurations.

We suggested that, independently of the type of baryons involved, each particle has associated with it a potential energy

$$U(N) = 3.2 \cdot 10^{-83} N^2 - 6.4 \cdot 10^{-5}, \tag{5.1}$$

where N is the baryon density. The formula is so chosen that at $N > 10^{40}$ cm^{-3} it exceeds the kinetic energy of the particles (including the rest energy), while at densities of a lower order it would coincide with the depth of the potential well in ordinary nuclear matter. Thus, according to Eq. (5.1), we have $U/E \approx 4$ at $N = 1.5 \cdot 10^{40}$ cm^{-3} and $U/E \approx 125$ at $N = 10^{41}$. Then $E = mc^2 (1-\beta^2)^{-1/2}$ is the upper cutoff of kinetic energy at the given density. As we see, formula (5.1) corresponds qualitatively to the real picture. However, in our calculations of stellar configurations, we disregarded the constant term in Eq. (5.1). This has no essential significance and consideration of that term in the work apparently led only to a slight decrease in mass. We repeat that our goal at this point is solely to trace out a qualitative picture of the state of affairs.

Bearing in mind Eq. (5.1), we now find for the equation of state [6]:

$$\rho = \rho_0 + NU(N); \; P = P_0 + N^2 \frac{\partial U}{\partial N}, \tag{5.2}$$

where ρ_0 and P_0 signify energy density and pressure for an ideal baryon gas. Equations (5.2) transform, in the case of low densities, to (3.1) and (3.2).

Substituting Eq. (5.2) into Eqs. (2.2') and (2.3'), we find

$$\frac{du}{dr} = r^2 \left\{ \sum_k \frac{1}{2} a_k \left(\frac{m_k}{m_n}\right)^4 (\operatorname{sh} t_k - t_k) + \frac{1}{K_n} \left[NU(N) + N_\pi \cdot m_\pi c^2 \right] \right\}. \tag{5.3}$$

$$\frac{dt_n}{\partial r} = -\frac{4}{r(r-2u)} \times$$

$$\times \frac{\sum_k \frac{1}{2} a_k \left(\frac{m_k}{m_n}\right)^4 \left(\operatorname{sh} t_k - 2 \operatorname{sh} \frac{t_k}{2}\right) + \frac{3}{4K_n} \left[NU(N) + N^2 \frac{dU}{dN} + N_\pi m_\pi c^2 \right]}{\sum_k \frac{1}{2} a_k \left(\frac{m_k}{m_n}\right)^4 \left(\operatorname{ch} t_k - 4 \operatorname{ch} \frac{t_k}{2} + 3\right) \frac{\partial t_k}{\partial t_n} + \frac{3}{K_n} \frac{d}{dN} \left(N^2 \frac{dU}{dN}\right) \frac{dN}{dt_n}} \tag{5.4}$$

$$\times \left\{ \frac{1}{3} r^3 \left[\sum_k \frac{1}{2} a_k \left(\frac{m_k}{m_n}\right)^4 \left(\operatorname{sh} t_k - 8 \operatorname{sh} \frac{t_k}{2} + 3t_k\right) + \frac{3}{K_n} N^2 \frac{dU}{dN} \right] + u(r) \right\}.$$

Integration of this set of equations is carried out numerically, and the task was facilitated by prior construction of graphs representing the right-hand members of Eqs. (5.3) and (5.4) as functions of parameter t_n.

TABLE 2. Several of the Most Important Parameters of Hyperonic Stars Consisting of a Real Baryon Gas

	Star						Hyperonic core of star			
	density at center		mass		coordinate radius		mass		coordinate radius	
$t_n(0)$	Baryon density $N(0)$, cm^{-3}	Matter density $4\pi\rho(0)$ g/cm^3	units of Eq. (2.5)	solar mass units	units of Eq. (2.5)	solar mass units	units of Eq. (2.5)	solar mass units	units of Eq. (2.5)	km
2.1	$6.40 \cdot 10^{38}$	$1.12 \cdot 10^{15}$	0.0770	0.715	0.892	12.2	—	—	—	—
2.2	$7.80 \cdot 10^{38}$	$1.44 \cdot 10^{15}$	0.0818	0.760	0.855	11.7	0.0073	0.0678	0,220	3.01
2.4	$1.25 \cdot 10^{39}$	$2.36 \cdot 10^{15}$	0.0926	0.860	0.740	10.0	0.038	0.353	0.346	4.74
2.6	$2.10 \cdot 10^{39}$	$4.31 \cdot 10^{15}$	0.108	1.007	0.607	8.32	0.079	0.734	0.393	5.38
3.0	$5.45 \cdot 10^{39}$	$1.67 \cdot 10^{16}$	0.111	1.028	0.394	5.40	0.104	0.966	0.325	4.45
3.1	$6.95 \cdot 10^{39}$	$3.56 \cdot 10^{16}$	0.102	0.947	0.370	5.07	0.098	0.910	0.305	4.18
3.2	$8.75 \cdot 10^{39}$	$5.75 \cdot 10^{16}$	0.0912	0.847	0.348	4.77	0.088	0.817	0.281	3.85
3.3	$1.08 \cdot 10^{40}$	$9.68 \cdot 10^{16}$	0.0836	0.777	0.341	4.67	0.080	0.743	0.268	3.67
3.4	$1.30 \cdot 10^{40}$	$1.38 \cdot 10^{17}$	0.0724	0.673	0.328	4.50	0.070	0.650	0.247	3.38
3.8	$2.60 \cdot 10^{40}$	$5.46 \cdot 10^{17}$	0.0717	0.666	0.349	4.79	0.0665	0.618	0.252	3.46
4.0	$3.50 \cdot 10^{40}$	$1.61 \cdot 10^{18}$	0.0722	0.670	0.358	4.90	0.0663	0.615	0.257	3.52
5.0	$1.26 \cdot 10^{41}$	$7.25 \cdot 10^{19}$	0.0738	0.686	0.360	4.93	0.068	0.631	0.259	3.55
5.32	$1.80 \cdot 10^{41}$	$2.01 \cdot 10^{20}$	0.0738	0.686	0.360	4.93	0.068	0.631	0.259	3.55
6.00	$3.53 \cdot 10^{41}$	$1.55 \cdot 10^{21}$	0.0738	0.686	0.360	4.93	0.068	0.631	0.259	3.55
7.00	$9.36 \cdot 10^{41}$	$2.88 \cdot 10^{22}$	0.0738	0.686	0.360	4.93	0.068	0.631	0.259	3.55
∞	∞	∞	0.0744	0.6905	0.3615	4.95	0.0692	0.643	0.2625	3.60

Results of the calculations are entered in Table 2 and graphed in Figs. 1, 2, 3. Curves 2a and 2b in Fig. 1 represent the mass of the star and the mass of its hyperon core as functions of $t_n(0)$. We see that the introduction of repulsive forces acting at close range leads to an appreciable increase in the possible values of stellar mass compared to the mass of configurations consisting of an ideal gas.

It might be asked to what extent this conclusion is essentially related to a special form of the potential of repulsive forces. Would it be possible for hyperon stars to acquire masses many times exceeding that of our sun, if the repulsive potential function be chosen appropriately?. This is answered in the negative for static configurations. The calculations which we have performed have convinced us that a reasonable choice of values for the radius of action of repulsion forces, independently of their intensity, lends no support to the notion of static configurations of large mass. This is shown in fact in the following section, using a model of an incompressible fluid as an example. The reason why it is impossible to accumulate large masses (compared to the solar mass) is the fact that, according to Einstein's theory of gravitation, the dimensions of static configurations in which density exceeds a certain density cutoff may not exceed a certain limit in size, since the gravitational radius may not exceed half the ordinary radius.

As curve 2a shows, the mass value has a clearly pronounced peak at $t_n(0) \approx 3$ ($N \approx 5.5 \cdot 10^{39}$ cm^{-3}). At $t_n(0) > 4$ ($N > 3.5 \cdot 10^{40}$ cm^{-3}) the stellar mass remains almost constant, i.e., is independent of density at the center of the configuration. It is interesting to note in this context that the bulk of the stellar matter is concentrated in the hyperon core for all these values of $t_n(0)$. The radius of the configuration varies to a very small extent in precisely the same manner for $t_n(0) > 4$.

Curves 2a and 2b in Fig. 2, describe the coordinate radius of the star and the coordinate radius of the hyperon core as functions of $t_n(0)$. As we see, the radius of the star at first varies inversely with the central density and then remains approximately constant. By comparing curves 1a and 2a, we find that in the case of a real gas the radius of the configuration is appreciably smaller than in the case of an ideal gas. The situation is the reverse for the radius of hyperon cores of the same configurations.

In Fig. 3 (curve 2), we have the relationship between mass and coordinate radius of a configuration consisting of a real gas. We see that the radius is not a single-valued function of mass for all values of the latter.

Note, however, that strictly speaking it is the number of baryons in a star, and not the mass, that is the basic parameter characterizing the configuration in our theory. Two configurations with the same number $\underline{n}$ of baryons may possess not only different radii but even different masses, since the mass defect in those configurations may be different. The configuration of lesser mass (larger mass defect) will be the more stable. A judicious choice of stable configurations could therefore be arrived at only on the basis of diagrams linking values of $\underline{n}$ and M.

To afford a comparison, configurations of a real neutron gas were also calculated, with the assumption that potential energy is in that case likewise defined by Eq. (5.1). The results obtained are indicated by full dots in Fig. 1. It is clear from inspection of the diagram that at a given central density the mass of hyperon-containing configurations is appreciably smaller than the mass of a hypothetical pure neutron star.

Note, finally, that even in the case of a nonideal gas, the model for which $t_n(0) = \infty$ was studied separately. The investigation was pursued in the same manner as in the preceding section with application to an ideal gas. It is readily seen that in this case, as $t_n \to \infty$, we have

$$P = 2\rho, \tag{5.5}$$

in other words, $z = 2$. From Eq. (5.5), we derive the asymptotic solution

$$\rho(r) = \frac{4}{17}\frac{1}{r^2}; \; u(r) = \frac{4}{17}r. \tag{5.6}$$

We assume this solution to be valid out to distance $r = 2 \cdot 10^{-4}$, where $t_n = 6.3$, $\rho(r) = 4.7 \cdot 10^5$; $P = 9.4 \cdot 10^5$. Furthermore, assuming these values to be the initial conditions, we carried out an integration to the point at which the density vanishes.

After much of our computational work was already behind us, we became informed of the recent contribution by Cameron [5]. This author calculated neutron configurations under the assumption that the energy density was

$$\varepsilon = 7.98 \cdot 10^9 \rho_m^{5/3} + 9.79 \cdot 10^{-6} \rho_m^{8/3} - 1.381 \cdot 10^5 \rho_m^2 \tag{5.7}$$

where $\rho_m = Nm_n$ is the density in g/cm^3, and $\varepsilon + Nm_n$ corresponds to our notation ρ. In Eqs. (5.7), the first term is the neutron kinetic energy, while the two following terms constitute the potential energy density.

The interaction between neutrons described by Eq. (5.7) enters into prominence at much longer range than the interaction described by our formula (5.1). The models arrived at by Cameron must therefore differ from the models calculated for an ideal Fermi gas, slightly larger than the models of neutron stars calculated on the basis of the interaction formula (5.1).

And in fact, the values for the masses of configurations at certain $t_n(0)$ values as reported by Cameron do significantly exceed these predicted by Eq. (1), which are graphed in Fig. 1 as black dots.

The mass values obtained by Cameron do seem too high to us, however, and it would be desirable to carry out more detailed quantitative comparisons.

VI. Model of an Incompressible Fluid

In the present section, we shall consider an "ideal gas" consisting of baryons of finite dimensions. When the particles are not in contact with each other the interaction energy vanishes. On the other hand, the particulars are absolutely hard and mutually impermeable.

We also assume that particles of all species have the same proper radius, although we shall take Lorentz contraction into account.

For a similar gas, some maximum density ρ_m must exist at which the space is packed to maximum density with particles. It may happen that this maximum density would be achieved in a given configuration at some distance R_0 from the center.

We shall have the following expression for the density:

$$\rho(r) = \begin{cases} \frac{1}{4\pi}\sum_k \frac{1}{2} a_k \left(\frac{m_k}{m_n}\right)^4 (\operatorname{sh} t_k - t_k) + m_\pi c^2 N_\pi, \\ \qquad \text{for } r > R_0, \\ \rho_m = \text{const}, \text{ for } r < P_0. \end{cases} \tag{6.1}$$

As for the pressure P, it is determined from the formula (3.2) for $r > r_0$, and for $r < R_0$ will be found to increase to some maximum value at the center.

Let us now find out at which values of the parameter t_n incompressibility will come about. It is clear that for this value of t_n we should have

$$\tau(t_n) \approx 1, \tag{6.2}$$

where τ is the total volume of particles enclosed in a unit volume of the star computed with Lorentz contraction taken into account. Of course we must include within τ such intervals as remain when particles are closely packed. For the volume τ we have

$$\tau = b \sum_k \frac{1}{2} a_k \int_0^{p_k} \frac{m_k c^2}{E_k(p)} \frac{p^2\,dp}{\pi^2 h^3}, \tag{6.3}$$

where b is the proper volume for one baryon.

Carrying out the indicated integration, we get

$$\tau(t_n) = b \sum_k \alpha_k \left(2sh\,\frac{t_k}{2} - t_k\right), \tag{6.4}$$

where we introduce the notation

$$\frac{a_k}{16\pi^2}\left(\frac{m_k c}{h}\right)^3 \equiv \alpha_k.$$

It thus follows from Eqs. (6.2) and (6.4) that incompressibility sets in starting with the value of the parameter t_n satisfying the equation

$$b \sum_k \alpha_k [2sh\,(t_k/2) - t_k] = 1. \tag{6.5}$$

To obtain the solution to Eq. (6.5), we must first have the value of the parameter b. We now assume that the value of b is equal to the quadrupled proper volume of the core. Bearing in mind the fact that this value must be of the order of 10^{-40} cm^{-3}, we assume, e.g., $b = 4.5 \cdot 10^{-40}$ cm^3. Substituting this value into Eq. (6.5), we now find for t_n^0

$$t_n^0 = 2.65.$$

We now determine the mass and radius of the central core consisting of incompressible matter. According to Eqs. (2.2')and (2.3'),we have

$$\frac{du}{dr} = 4\pi r^2 \rho_m;$$
$$\frac{dP}{dr} = -\frac{P+\rho_m}{r(r-2u)}(4\pi r^3 P + u), \qquad (6.6)$$

where ρ_m is indeed our maximum density, equal to 0.63, at which incompressibility sets in. The value of ρ_m referred to was found from the equation

$$\rho_m = \frac{1}{4\pi}\sum_k \frac{1}{2} a_k (\operatorname{sh} t_k^0 - t_k^0), \qquad (6.7)$$

where $t_k^0 \equiv t_k(t_n^0)$ are constant numbers.

Integrating Eq. (6.5) from zero to r and remembering that u (0) = 0, we obtain

$$u(r) = \frac{4\pi}{3}\rho_m r^3. \qquad (6.8)$$

Substituting Eq. (6.8) into (6.6) and integrating from the center of the star to the boundary of the incompressible sphere, we obtain for the coordinate radius of that sphere

$$R_0 = \left\{\frac{2}{8\pi\rho_m}\left[1 - \left(\frac{3P_m+\rho_m}{P_m+\rho_m}\frac{P_c+\rho_m}{3P_c+\rho_m}\right)^2\right]\right\}^{1/2} \qquad (6.9)$$

where P_c and P_m are respectively the pressure at the center of the star and the pressure at the surface of the incompressible sphere. Substituting $t_n^0 = 2.65$ into Eqs. (3.2'),we find $\rho_m = 0.625$; $P_m = 0.0342$.

From Eq. (6.9) and (6.8), we find the mass of the entire incompressible sphere:

$$u(R_0) = \frac{0.613}{\sqrt{4\pi\rho_m}}\left[1 - \left(\frac{3P_m+\rho_m}{P_m+\rho_m}\frac{P_c+\rho_m}{3P_c+\rho_m}\right)^2\right]^{3/2}. \qquad (6.10)$$

Substituting $\rho_m = 0.63$ and $P_m = 0.034$ into Eqs. (6.9) and (6.10), we arrive at

$$R_0 = 0.436\left[1 - 1.22\left(\frac{P_c+\rho_m}{3P_c+\rho_m}\right)^2\right]^{1/2};$$
$$u(R_0) = 0.22\left[1 - 1.22\left(\frac{P_c+\rho_m}{3P_c+\rho_m}\right)^2\right]^{3/2}. \qquad (6.11)$$

Considering, in particular, configurations for which $P_c > \rho_m$, we obtain, for the mass and radius of an incompressible sphere at $t_n^0 = 2.65$

$$u(R_0) = 0.175; R = 0,406. \qquad (6.12)$$

Moreover, assuming that the equation of state is determined from formulas (3.1) and (3.2), and proceeding from initial conditions (6.12), we may integrate Eqs. (2.2) and (2.3) to the periphery of the star, i.e., to that distance R at which

$$\rho(R) = P(R) = 0,$$

as a result of which we obtain

$$R = 6 \text{ km}; \ M = 1.7\odot. \qquad (6.13)$$

Choosing for the volume b a large value, we might end up with a large configuration mass.

Summary

The calculations carried out demonstrate that the masses of equilibrium configurations of a degenerate nonrotating baryon gas,in a case where we assume the gas to be an ideal gas, are of the order of a half of a solar mass, and that the radii reach out several kilometers.

The masses of the degenerate configurations calculated for a real Fermi gas of baryons under the assumption that repulsive forces are active at close range between baryons, are appreciably larger than the masses of configurations of an ideal gas. However, small masses of the order of a solar mass are also obtained in this case. Even if we alter the law of repulsion assumed to prevail, we would still be unable to obtain configurations with masses far exceeding, in order of magnitude, a solar mass.

The sizes and masses of the outer regions of a baryon star, i.e., the neutron layer or proton-electron layer, comprise a small part of the total mass and size of the star at fairly high central densities. The bulk of the star's mass in those cases goes into the hyperon core. These configurations are therefore conveniently termed hyperon configurations.

Configurations with slightly lower central densities ($\rho < 10^{15}$ g/cm^3) lack hyperon cores and consist entirely of neutrons.

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